The Richard Stockton College of New Jersey
Mathematical Mayhem 2013
Individual Round

March 23, 2013

Name: ____

High School: ____

Instructions:

- This round consists of **18** problems worth a total of **80** points, made up of 8 Appetizers worth 3 points each, 7 Entrées worth 5 points each, and 3 Desserts worth 7 points each.
- Each of the 18 problems is multiple choice, and each problem comes with **5** possible answers.
- For each problem, **circle the best answer**.
- You are not required to show any work this round.
- No calculators are permitted.
- This round is **75 minutes** long. **Good Luck!**

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Problem #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	Total
Points Earned																			

♣ Appetizers ♣

Problem 1. Which of the following is not equal to $\frac{15}{4}$?

(A.) 3.75 (B.) $\frac{14+1}{3+1}$ (C.) $\frac{3}{4}+3$ (D.) $\frac{21}{4}-\frac{5}{4}-\frac{1}{4}$ (E.) $\boxed{\frac{5}{4}\times\frac{3}{4}}$

Problem 2. Four friends go fishing one day and bring home a total of 11 fish. If each person caught at least one fish, then which of the following must be true?

I. Somebody caught exactly 3 fish

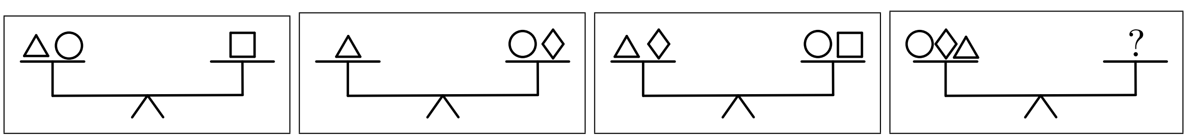
II. Somebody caught fewer than 3 fish

III. Somebody caught more than 3 fish

(A.) Only I (B.) $\boxed{\text{Only II}}$ (C.) Only III (D.) Only I and II (E.) Only II and III

Problem 3. A rectangular sheet of paper measures 25 cm $\times$ 9 cm. The dimensions of a square sheet of paper with the same area are
(A.) $\boxed{15 \text{ cm} \times 15 \text{ cm}}$ (B.) 17 cm $\times$ 17 cm (C.) 8 cm $\times$ 8 cm (D.) 16 cm $\times$ 16 cm (E.) 34 cm $\times$ 34 cm

Problem 4. All of the scales shown below are balanced. One possible replacement for the "?" is
(A.) $\boxed{\square\diamond}$ (B.) $\diamond\triangle$ (C.) $\bigcirc\square$ (D.) $\triangle\square$ (E.) $\triangle\bigcirc$



Direct substitution from the first scale demonstrates the solution. Using the first three scales, working in units of the weight of the circle, we can see that $\bigcirc = 1$, $\diamond = 2$, $\triangle = 3$ and $\square = 4$. Then the left side of the fourth scale has a total weight of 6, and, of the answers, only $\square\diamond$ has a weight of 6.

Problem 5. If $f(x) = x^2 + 1$, evaluate $f(f(f(1)))$.
(A.) 1 (B.) 4 (C.) 5 (D.) $\boxed{26}$ (E.) 676

Problem 6. The number $\frac{(\sqrt{18}-\sqrt{2})^2}{(\sqrt[3]{16}-\sqrt[3]{2})^3}$ =?

(A.) $\sqrt{2}/2$ (B.) 1 (C.) $\sqrt{2}$ (D.) 2 (E.) $\boxed{4}$

$$\frac{\left(\sqrt{18}-\sqrt{2}\right)^2}{\left(\sqrt[3]{16}-\sqrt[3]{2}\right)^3} = \frac{(2\sqrt{2})^2}{(\sqrt[3]{2})^3} = 4$$

Problem 7. If n is a real number, then the simultaneous system of equations below has no solution if and only if n is equal to
(A.) $\boxed{-1}$ (B.) 0 (C.) 1 (D.) 0 or 1 (E.) $\frac{1}{2}$

$$\begin{aligned} nx + y &= 1 \\ x + ny &= 1 \end{aligned}$$

The determinant of the two linear equations is $n^2 - 1$. Only when $n = 1$ or $n = -1$, the determinant is zero. When $n = 1$, there are infinitely many solutions for the two equations. When $n = -1$, the two equations have no solution.

Problem 8. Amy, Bob, and Chris each took a 6-question true-false exam. Their answers to the six questions in order were Amy:FFTTTT, Bob:TFFTTT, and Chris:TTFFTT. Amy and Bob each got 5 right. What is the most that Chris could have gotten right?
(A.) 1 (B.) 2 (C.) $\boxed{3}$ (D.) 4 (E.) 5

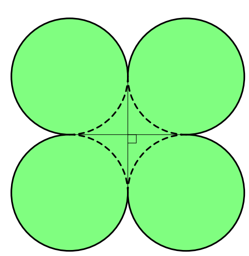
The correct answers must have been either FFFTTT or TFTTTT. Either way, Chris got 3 right.

♢ Entrées ♢

Problem 9. Suppose that x and y are four digit numbers containing exactly one 0, one 1, one 2 and one 3. The leftmost digits of x and y cannot be 0. What is the maximum possible absolute difference $|x - y|$?
(A.) 0 (B.) 1998 (C.) $\boxed{2187}$ (D.) 2889 (E.) 3087

$3210 - 1023 = 2187$

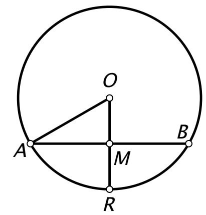
Problem 10. The four leaf clover shown below is made up of four circles of radius 1, each of which is tangent to two others, and a central wedge that is bounded by the circles. What is the area of the clover?
(A.) 4π (B.) $\boxed{4 + 3\pi}$ (C.) 16 (D.) $3 + 4\pi$ (E.) 13.2



Breaking the wedge into four identical pieces (by the dotted lines in the figure, and gluing those pieces around a circle produces a square. The area of the clover is the area of 3 circles of radius 1 and a square of side length 2, or $4 + 3\pi$.

Problem 11. A chord which is the perpendicular bisector of a radius of length 12 in a circle, has length
(A.) $3\sqrt{3}$ (B.) 27 (C.) $6\sqrt{3}$ (D.) $\boxed{12\sqrt{3}}$ (E.) none of these

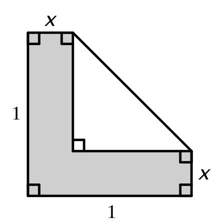
Let O be the center of the circle, and let OR and AB be the radius and the chord which are perpendicular bisectors of each other at M respectively. By the Pythagorean theorem, we have $AM^2 = OA^2 - OM^2 = 12^2 - 6^2 = 108$. Hence $AM = 6\sqrt{3}$. The length of the chord is $12\sqrt{3}$.



Problem 12. How many integers x in $\{1, 2, 3, \ldots 99, 100\}$ are there so that $x^2 + x^3$ is the square of an integer?
(A.) 6 (B.) 7 (C.) 8 (D.) $\boxed{9}$ (E.) 10

Note $x^3 + x^2 = x^2(x + 1)$, and so $x^3 + x^2$ is a perfect square if and only if $x + 1$ is a perfect square. The values $x = 3, 8, 15, 24, 35, 48, 63, 80, 99$ make $x^3 + x^2$ a perfect square.

Problem 13. In the figure below, if the shaded area equals the area of the triangle, what is the length x?
(A.) $1/2$ (B.) $\sqrt{3}/3$ (C.) 1 (D.) $1 + \sqrt{6}/3$ (E.) $\boxed{1 - \sqrt{6}/3}$



Note $(1/2)(1 - x)(1 - x) = x^2 + 2x(1 - x)$. Applying the quadratic equation yields $x = 1 \pm \sqrt{(}6)/3$, and since $x < 1$ the only applicable solution is $x = 1 - \sqrt{6}/3$.

Problem 14. When one ounce of water is added to a mixture of acid and water, the new mixture is 20% acid. When one ounce of acid is added to the new mixture, the result is $33\frac{1}{3}$% acid. The percentage of acid in the original mixture is
(A.) 22% (B.) 24% (C.) $\boxed{25\%}$ (D.) 30% (E.) $33\frac{1}{3}$%

Let x and y be the number of ounces of water and of acid, respectively, in the original solution. After the addition of 1 ounce of water, there are y ounces of acid and $x + y + 1$ ounces of solution. After adding 1 ounce of acid, there are $y + 1$ ounces of acid and $x + y + 2$ ounces of solution. We have

$$\begin{aligned} \frac{y}{x+y+1} &= \frac{1}{5}, \\ \frac{y+1}{x+y+2} &= \frac{1}{3}. \end{aligned}$$

Solving the above equations, we get $x = 3$ and $y = 1$. Hence $\frac{y}{x+y} = \frac{1}{1+3} = 25\%$.

Problem 15. What is the sum of the digits in the decimal representation of $(10^{10}+1)^2$?
(A.) 1 (B.) 2 (C.) $\boxed{4}$ (D.) 8 (E.) none of these

By looking at small powers of 10, observe the following pattern:

$$(10000000001)^2 = 1(\text{zeros})2(\text{zeros})1.$$

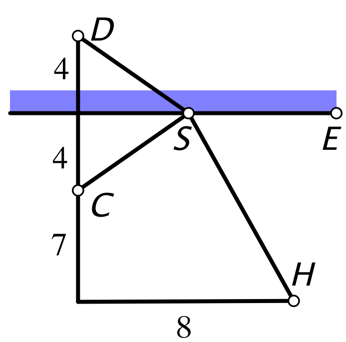
♡ Desserts ♡

Problem 16. Beginning with 1, write all positive integers successively, beginning as 12345678910111213 . . .. What digit appears in the 2013th position?
(A.) 3 (B.) 4 (C.) 5 (D.) 6 (E.) $\boxed{7}$

All one and two digit numbers require 189 digits $2013 - 189 = 1824$, and $1824/3 = 608$. This is how many three digit numbers we need to count or find. Since $608 + 99 = 707$, the units digit is 7.

Problem 17. A cowboy is 4 miles south of a stream which flows due east. He is also 8 miles west and 7 miles north of his cabin. He wishes to water his horse at the stream and return home. The shortest distance (in miles) he can travel and accomplish this is
(A.) $4+\sqrt{185}$ (B.) 16 (C.) $\boxed{17}$ (D.) 18 (E.) $\sqrt{32}+\sqrt{137}$

In the graph, let S be any point on the stream SE, and C, H and D be the position of the cowboy, his cabin, and the point 8 miles north of C, respectively. The distance $CS + SH$ = the distance $DS + SH$, which is the smallest when DSH is straight. $CSH = DSH = \sqrt{8^2 + 15^2} = 17$ miles.



Problem 18. It's between 5:30 and 6 pm, and the minute and hour hands are perpendicular. What time is it (to the nearest minute)?
(A.) 5:41 pm (B.) 5:42 pm (C.) 5:43 pm (D.) $\boxed{\text{5:44 pm}}$ (E.) 5:45 pm

Taking "minutes after 5 pm" as our unit of angle measure, at x minutes after 5 pm the hour hand points to $25 + \frac{5x}{60}$. Between 5:30 and 6 pm, the minute and hour hands are perpendicular once, when the position of the minute hand is 15 minutes greater than the position of the hour hand. That is, they will be perpendicular x minutes after 5 pm, where x is the solution to:

$$25 + \frac{5x}{60} + 15 = x.$$

Solving for x gives $44\frac{4}{11} \approx 44$, making the answer 5:44 pm.